Criteria for k-positivity of linear maps

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Abstract. We study k-positive maps on operators. Short proofs are given to different positivity criteria. Special attention is given to positive maps arising in the study of quantum information science. Results of other researchers are extended and improved. New classes of positive maps are constructed. Some open questions are answered.

1. Introduction

Denote by $B(H, K)$ the set of bounded linear operators from the Hilbert space H to the Hilbert space K, and write $B(H, K) = B(H)$ if $H = K$. Let $B(H)^+$ be the set of positive semidefinite operators in $B(H)$. If H and K have dimensions n and m respectively, we identify $B(H, K)$ with the set $M_{m,n}$ of $m \times n$ matrices, and write $M_{n,n} = M_n$, and $B(H)^+ = M_n^+$.

A linear map $L : B(H) \to B(K)$ is positive if $L(B(H)^+) \subseteq B(K)^+$. For a positive integer k, the map L is k-positive if the map $I_k \otimes L : M_k(B(H)) \to M_k(B(K))$ is positive, where $(I_k \otimes L)(A) = (L(A_{ij}))$ for any $A = (A_{ij})_{1 \le i,j \le k}$ with $A_{ij} \in B(H)$. A map is completely positive if it is k-positive for every positive integer k. The study of positive maps has been the central theme for many pure and applied topics; for example, see [4, 15, 18, 19, 20]. In particular, the study has attracted a lot of attention of physicists working in quantum information science in recent decades, because positive linear maps can be used to distinguish entanglement of quantum states (see [12]). There is considerable interest in finding positive maps that are not completely positive, which can be applied to detect entangled states (see, for example, [1, 3, 6, 7, 8, 9, 14, 16, 17, 22] and the references therein). Completely positive linear maps have been well studied by researchers. However, the structure of positive linear maps is still unclear even for the finite dimensional case ([5, 10, 15, 21]).

In this paper, we give a brief summary of existing criteria of k-positive maps on operators for convenient reference. Short proofs are given to these different positivity criteria. Special attention is given to positive maps arising in the study of quantum information science. Furthermore, some of the existing results are extended and improved and some open problems are addressed.

The paper is organized as follows. Sections 2 and 3 summarize some basic known criteria for the different types of k-positive maps and several new criteria for elementary operators by using k-numerical range of operators are presented (Propositions 2.1-2.2 and 3.1-3.2). In Section 4, we extend and generalize the results of Chruściński and Kossakowski in [6] (see Propositions 4.2 - 4.3) by the tools introduced in Section 3. In Section 5, we discuss a family of positive maps, called D-type positive maps, which is a generalization of Choi's maps and was often used in quantum information theory. We give a necessary and sufficient condition for such maps to be k-positive (Proposition 5.1 and Corollary 5.2). Section 6 is devoted to illustrate the application of results in Section 5 to the construction of new positive D-type linear maps (Examples 6.1, 6.6 and 6.7, Propositions 6.2 and 6.3). In Section 7, we consider the decomposability of positive

linear maps, propose a new class of decomposable positive maps, and answer an open problem (Proposition 7.2). Section 8 is a short conclusion.

2. Basic criteria

In this section, we present several equivalent conditions of k-positivity and provide short and elementary proofs of them. Some of the conditions were presented in [4, 15]. We also present some new results. In the following, a vector of H will be denoted by $|x\rangle$ and $\langle x|$ is defined to be the dual vector of the vector $|x\rangle$ in the dual space of H.

Proposition 2.1 *Suppose $L : B(H) \to B(K)$ is a linear map continuous under strong operator topology. The following are equivalent.*

(a) *L is k-positive, i.e., $I_k \otimes L$ is positive.*

(b) *$(I_k \otimes L)(P)$ is positive semi-definite for any rank one orthogonal projection $P \in M_k(B(H))$.*

(c) *For any (orthonormal) subset $X = \{|x_1\rangle, \ldots, |x_k\rangle\} \subseteq H$, the operator matrix defined by $L_X = (\, L(|x_i\rangle\langle x_j|)\,)_{1\le i,j\le k}$ is positive semi-definite.*

Proof. The implications (a) $\iff$ (b) $\Longrightarrow$ (c) are clear because the set of finite rank positive operator is strongly dense in $B(H)^+$ and L is strongly continuous. To prove (c) $\Rightarrow$ (b), one only needs to check the condition for orthonormal set $\{|x_1\rangle, \ldots, |x_k\rangle\} \subseteq H$. For every $|z\rangle \in H^{\oplus k}$, write $|z\rangle = \sum_{i=1}^k |e_i\rangle \otimes |z_i\rangle$ where $|z_i\rangle \in H$ and $\{|e_i\rangle\}_{i=1}^k$ is the canonical basis of $\mathbf{C}^k$, and define the finite rank operator $Z = \sum_{i=1}^k |z_i\rangle\langle e_i|$. Consider the singular value decomposition (a.k.a. the Schmidt decomposition in the context of quantum information science) of $Z = \sum_{i=1}^k |y_i\rangle\langle x_i|$, one can get a decomposition $|z\rangle = \sum_{j=1}^k |y_j\rangle \otimes |x_j\rangle$, where $\{|y_1\rangle, \ldots, |y_k\rangle\}$ is an orthogonal set in $\mathbf{C}^k$, and $\{|x_1\rangle, \ldots, |x_k\rangle\}$ is an orthonormal set in H. Let $Y = \left(\sum_{i=1}^k |y_i\rangle\langle e_i|\right) \otimes I_H$. Then

$$(I_k \otimes L)(|z\rangle\langle z|) = (I_k \otimes L)\left(\left(\sum_{j=1}^k |y_j\rangle|x_j\rangle\right)\left(\sum_{j=1}^k \langle x_j|\langle y_j|\right)\right) = Y L_X Y^\dagger$$

is positive semi-definite by assumption. □

Suppose $L : M_n \to B(K)$ is a linear map. Let $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ be the standard basis for M_n. The Choi matrix $C(L)$ is the operator matrix with $(L(E_{ij}))_{1\le i,j\le n}$. Clearly, there is a one-one correspondence between a linear map L and the Choi matrix $C(L)$. One can use the Choi matrix to determine whether the map L is k-positive.

Proposition 2.2 *Let $L : M_n \to B(K)$ and $1 \le k \le n$. The following are equivalent.*

(a) L *is k-positive.*

(b) $\langle x|C(L)|x\rangle \ge 0$ *for all* $|x\rangle = \sum_{p=1}^{k} |y_p\rangle \otimes |z_p\rangle$ *with* $|y_p\rangle \otimes |z_p\rangle \in \mathbf{C}^n \otimes K$.

(c) $(I_n \otimes P)C(L)(I_n \otimes P)$ *is positive semi-definite for any rank-k orthogonal projection* $P \in B(K)$.

Proof. (a) $\Leftrightarrow$ (b) : First consider the case $k = 1$. Let $\{|e_i\rangle : 1 \le i \le n\}$ be the canonical basis for $\mathbf{C}^n$. Then $C(L) = \sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes L(|e_i\rangle\langle e_j|)$. We have

$$
\begin{aligned}
& L \ge 0 \\
\Leftrightarrow\ & L(|y\rangle\langle y|) \ge 0 \ \text{ for all } |y\rangle \in \mathbf{C}^n \\
\Leftrightarrow\ & (\langle y| \otimes I_K)C(L)(|y\rangle \otimes I_K) \ge 0 \ \text{ for all } |y\rangle \in \mathbf{C}^n \\
\Leftrightarrow\ & \langle z|\,(\,(\langle y \otimes I_K)C(L)(|y\rangle \otimes I_K)\,)\,|z\rangle \ge 0 \ \text{ for all } |y\rangle \in \mathbf{C}^n,\ |z\rangle \in K \\
\Leftrightarrow\ & (\langle y|\langle z|)C(L)(|y\rangle|z\rangle)) \ge 0 \ \text{ for all } |y\rangle \in \mathbf{C}^n,\ |z\rangle \in K \\
\Leftrightarrow\ & \langle x|C(L)|x\rangle \ge 0 \ \text{ for all } |x\rangle = |y\rangle|z\rangle \text{ with } |y\rangle \in \mathbf{C}^n,\ |z\rangle \in K.
\end{aligned}
$$

For general $k > 1$, let $\{|f_p\rangle : 1 \le p \le k\}$ be the canonical basis for $\mathbf{C}^k$. Then

$$
C(I_k \otimes L) = \sum_{p,q=1}^{k} \sum_{i,j=1}^{n} (|f_p\rangle\langle f_q| \otimes |e_i\rangle\langle e_j|) \otimes (|f_p\rangle\langle f_q| \otimes L(|e_i\rangle\langle e_j|))\,. \tag{1}
$$

Note that every $|\tilde{y}\rangle \in \mathbf{C}^k \otimes \mathbf{C}^n$ (respectively, $|\tilde{z}\rangle \in \mathbf{C}^k \otimes K$) has the form

$$
|\tilde{y}\rangle = \sum_{r=1}^{k} |f_r\rangle \otimes |y_r\rangle \qquad \left(\text{respectively, } |\tilde{z}\rangle = \sum_{s=1}^{k} |f_s\rangle \otimes |z_s\rangle \right), \tag{2}
$$

where $|y_r\rangle \in \mathbf{C}^n$, $|z_s\rangle \in K$, $1 \le r, s \le k$. Now, applying the above result to $I_k \otimes L : M_k \otimes M_n \to M_k \otimes B(K)$, by (1) and (2), we have

$$
\begin{aligned}
& I_k \otimes L \ge 0 \\
\Leftrightarrow\ & (\langle\tilde{y}| \otimes \langle\tilde{z}|)C(I_k \otimes L)(|\tilde{y}\rangle \otimes |\tilde{z}\rangle) \ge 0 \text{ for all } |\tilde{y}\rangle \in \mathbf{C}^k \otimes \mathbf{C}^n \text{ and } |\tilde{z}\rangle \in \mathbf{C}^k \otimes K \\
\Leftrightarrow\ & \left(\sum_{r,s=1}^{k} \langle f_r|\langle y_r|\langle f_s|\langle z_s| \right) \left(\sum_{p,q=1}^{k} \sum_{i,j=1}^{n} |f_p\rangle\langle f_q| \otimes |e_i\rangle\langle e_j| \otimes |f_p\rangle\langle f_q| \otimes L(|e_i\rangle\langle e_j| \right) \\
& \left(\sum_{r',s'=1}^{k} |f_{r'}\rangle|y_{r'}\rangle|f_{s'}\rangle|z_{s'}\rangle \right) \ge 0 \text{ for all } |y_r\rangle \in \mathbf{C}^n \text{ and } |z_s\rangle \in K, 1 \le r, s \le k
\end{aligned}
$$

$$\Leftrightarrow \left(\sum_{p=1}^{k} \langle y_p|\langle z_p|\right)\left(\sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes L(|e_i\rangle\langle e_j|)\right)\left(\sum_{q=1}^{k} |y_q\rangle|z_q\rangle\right) \geq 0$$

$$\text{for all } |y_p\rangle \in \mathbf{C}^n \text{ and } |z_p\rangle \in K, 1 \leq p \leq k$$

$$\Leftrightarrow \langle x|C(L)|x\rangle \geq 0 \text{ for all } |x\rangle = \sum_{p=1}^{k} |y_p\rangle|z_p\rangle \text{ with } |y_p\rangle|z_p\rangle \in \mathbf{C}^n \otimes K.$$

(b) $\Leftrightarrow$ (c) : Suppose (c) holds. Given $|x\rangle = \sum_{p=1}^{k} |y_p\rangle \otimes |z_p\rangle$, where $|y_p\rangle \in \mathbf{C}^n$ and $|z_p\rangle \in K$, $1 \leq p \leq k$, let P be the orthogonal projection to the subspace spanned by $\{|z_p\rangle : 1 \leq p \leq k\}$. Then $(I_k \otimes P)|x\rangle = |x\rangle$. Therefore,

$$\langle x|C(L)|x\rangle = (\langle x|(I_k \otimes P))C(L)((I_k \otimes P)|x\rangle) = \langle x| \left((I_k \otimes P)C(L)(I_k \otimes P)\right)|x\rangle \geq 0.$$

Conversely, suppose (b) holds. Let P be an orthogonal projection in K with rank k and $\{|z_p\rangle : 1 \leq p \leq k\}$ be an orthonormal basis of the range space of P. For every $|w\rangle \in \mathbf{C}^n \otimes K$, there exist $|y_p\rangle \in \mathbf{C}^n$, $1 \leq p \leq k$ such that $(I_n \otimes P)|w\rangle = \sum_{p=1}^{k} |y_p\rangle \otimes |z_p\rangle$. We have

$$\langle w|(I_n \otimes P)C(L)(I_n \otimes P)|w\rangle = \left(\sum_{p=1}^{k} \langle y_p|\langle z_p|\right) C(L) \left(\sum_{p=1}^{k} |y_p\rangle|z_p\rangle\right) \geq 0.$$

Hence, $(I_n \otimes P)C(L)(I_n \otimes P) \geq 0$. □

3. Elementary operators

An operator $L : B(H) \to B(K)$ is called an elementary operator if it has the form

$$L(X) = \sum_{j=1}^{k} A_j X B_j^\dagger$$

for some $A_1, \ldots, A_k, B_1, \ldots, B_k \in B(H, K)$ [15]. If H and K are finite dimensional, then every linear map is elementary. Since we are interested in positive linear map, we focus on linear map which maps self-adjoint operators to self-adjoint operators. Thus, for any self-adjoint X,

$$\sum_{j=1}^{k} A_j X B_j^\dagger = L(X) = L(X)^\dagger = \sum_{j=1}^{k} B_j X A_j^\dagger.$$

As a result, for any self-adjoint X, we get

$$2L(X) = \sum_{j=1}^{k}(A_j X B_j^\dagger + B_j X A_j^\dagger) = \sum_{j=1}^{k}(A_j + B_j)X(A_j + B_j)^\dagger - \sum_{j=1}^{k}(A_j X A_j^\dagger + B_j X B_j^\dagger).$$

By linearity, the above equation is true for all $X \in B(H)$. Thus we will focus on elementary operators of the form

$$L(X) = \sum_{j=1}^{p} C_j X C_j^\dagger - \sum_{j=1}^{q} D_j X D_j^\dagger.$$

Hou [15] gave a condition for an elementary operator in the above form to be k-positive. In this section, we will extend those results by Proposition 2.1 in the following.

Proposition 3.1 *Suppose $L : B(H) \to B(K)$ has the form*

$$X \mapsto \sum_{r=1}^{p} C_r X C_r^\dagger - \sum_{s=1}^{q} D_s X D_s^\dagger \tag{3}$$

with $C_1, \dots, C_p, D_1, \dots, D_q \in B(H, K)$. Then

$$(I_k \otimes L)(X) = \sum_{r=1}^{p} (I_k \otimes C_r) X (I_k \otimes C_r^\dagger) - \sum_{s=1}^{q} (I_k \otimes D_s) X (I_k \otimes D_s^\dagger).$$

Moreover, the following are equivalent.

(a) *L is k-positive, i.e., $I_k \otimes L$ is positive.*

(b) *$\sum_{r=1}^{p} (I_k \otimes C_r) X (I_k \otimes C_r^\dagger) - \sum_{s=1}^{q} (I_k \otimes D_s) X (I_k \otimes D_s^\dagger) \in M_k(B(K))^+$ for any rank one orthogonal projection $X \in M_k(B(H))$.*

(c) *For any (orthonormal) subset $\{|x_1\rangle, \dots, |x_k\rangle\} \subseteq H$, $\sum_{i,j=1}^{k} E_{ij} \otimes L(|x_i\rangle\langle x_j|)$ is positive semi-definite, equivalently,*

$$\sum_{r=1}^{p} \sum_{i,j=1}^{k} E_{ij} \otimes C_r |x_i\rangle\langle x_j| C_r^\dagger \geq \sum_{s=1}^{q} \sum_{i,j=1}^{k} E_{ij} \otimes D_s |x_i\rangle\langle x_j| D_s^\dagger.$$

(d) *For any $|x\rangle \in \mathbf{C}^k \otimes H$, there is an $q \times p$ matrix T_x with operator norm $\|T_x\| \leq 1$ such that*

$$\begin{pmatrix} I_k \otimes D_1 \\ I_k \otimes D_2 \\ \vdots \\ I_k \otimes D_q \end{pmatrix} |x\rangle = (T_x \otimes I_K) \begin{pmatrix} I_k \otimes C_1 \\ I_k \otimes C_2 \\ \vdots \\ I_k \otimes C_p \end{pmatrix} |x\rangle.$$

Proof. The equivalence of (b) and (c) follows from Proposition 2.1 and the special form of L. For the equivalence of (a), (b) and (d), see [15]. □

Recall that, for a linear operator $A \in B(H)$ and a positive integer $k \leq \dim H$, the k-numerical range of A is defined by

$$W_k(A) = \left\{ \sum_{j=1}^{k} \langle x_j | A | x_j \rangle : \{|x_1\rangle, \dots, |x_k\rangle\} \text{ is an orthonormal set in } H \right\}.$$

If $\dim H = n < \infty$, and A is Hermitian with eigenvalues $a_1 \geq \cdots \geq a_n$, then

$$W_k(A) = \left[\sum_{j=1}^{k} a_{n-j+1}, \sum_{j=1}^{k} a_j \right].$$

For the details of k-numerical ranges, see [2].

The following proposition gives the relation between k-numerical ranges and k-positivity of elementary operators.

Proposition 3.2 *Suppose $L : M_n \to B(K)$ has the form (3).*

(a) *If L is k-positive, then $W_k\left(\sum_{r=1}^{p} C_r^\dagger C_r - \sum_{s=1}^{q} D_s^\dagger D_s\right) \subseteq [0, \infty)$.*

(b) *If for any unit vectors $|u\rangle = (u_1, \ldots, u_p)^t \in \mathbf{C}^p$ and $|v\rangle = (v_1, \ldots, v_q)^t \in \mathbf{C}^q$,*

$$\min W_k \left((\sum_r u_r C_r)^\dagger (\sum_r u_r C_r) \right) \geq \max W_k \left((\sum_s v_s D_s)^\dagger (\sum_s v_s D_s) \right), \qquad (4)$$

then L is k-positive.

Here, $\min S$ *and* $\max S$ *denote the minimum and maximum value of a subset S of real number.*

Proof. Denote $\Gamma_{n,k}$ by the set of vectors $|\mathbf{x}\rangle = \begin{pmatrix} |x_1\rangle \\ \vdots \\ |x_k\rangle \end{pmatrix}$ such that $\{|x_1\rangle, \ldots, |x_k\rangle\} \subseteq \mathbf{C}^n$ is an orthonormal set.

If L is k-positive, then $(I_k \otimes L)(|\mathbf{x}\rangle\langle\mathbf{x}|)$ is positive semi-definite for every $|\mathbf{x}\rangle \in \Gamma_{n,k}$. Taking trace, we see that

$$0 \leq \sum_{j=1}^{k} \operatorname{tr} \left(\sum_r C_r |x_j\rangle\langle x_j| C_r^\dagger - \sum_s D_s |x_j\rangle\langle x_j| D_s^\dagger \right) = \sum_{j=1}^{k} \langle x_j| \left(\sum_r C_r^\dagger C_r - \sum_s D_s^\dagger D_s \right) |x_j\rangle.$$

The result (a) follows.

For (b), suppose (4) holds for any unit vectors $|u\rangle = (u_1, \ldots, u_p)^t \in \mathbf{C}^p$ and $|v\rangle = (v_1, \ldots, v_q)^t \in \mathbf{C}^q$. For $|\mathbf{x}\rangle \in \Gamma_{n,k}$, let

$$\tilde{C}_{\mathbf{x}} = \begin{pmatrix} C_1|x_1\rangle & \cdots & C_p|x_1\rangle \\ \vdots & \ddots & \vdots \\ C_1|x_k\rangle & \cdots & C_p|x_k\rangle \end{pmatrix} \quad \text{and} \quad \tilde{D}_{\mathbf{x}} = \begin{pmatrix} D_1|x_1\rangle & \cdots & D_q|x_1\rangle \\ \vdots & \ddots & \vdots \\ D_1|x_k\rangle & \cdots & D_q|x_k\rangle \end{pmatrix}.$$

We will show that $\tilde{C}_{\mathbf{x}}\tilde{C}_{\mathbf{x}}^\dagger - \tilde{D}_{\mathbf{x}}\tilde{D}_{\mathbf{x}}^\dagger$ is positive semi-definite, or equivalently, for any unit vector $|y\rangle \in \mathbf{C}^{kp}$,

$$\|\langle y|\tilde{C}_{\mathbf{x}}\|^2 \geq \|\langle y|\tilde{D}_{\mathbf{x}}\|^2.$$

Denote by $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_{\min\{m,n\}}(A)$ be the singular values of $A \in M_{m,n}$. Note that there is $|\tilde{\mathbf{x}}\rangle \in \Gamma_{n,k}$ so that $\tilde{C}_{\tilde{\mathbf{x}}}$ has the smallest p-th singular value $\sigma_p(\tilde{C}_{\tilde{\mathbf{x}}})$ among all choices of $|\mathbf{x}\rangle \in \Gamma_{n,k}$.

$$\|\langle y|\tilde{C}_{\mathbf{x}}\| \ge \sigma_p(\tilde{C}_{\mathbf{x}}) \ge \sigma_p(\tilde{C}_{\tilde{\mathbf{x}}}).$$

Moreover, there is a unit vector $|\tilde{u}\rangle = (\tilde{u}_1, \ldots, \tilde{u}_p)^t \in \mathbf{C}^p$ such that

$$\begin{aligned}
(\sigma(\tilde{C}_{\tilde{\mathbf{x}}}))^2 = \|\tilde{C}_{\tilde{\mathbf{x}}}|\tilde{u}\rangle\|^2 &= \left\| \begin{pmatrix} (\sum_r \tilde{u}_r C_r)|\tilde{x}_1\rangle \\ \vdots \\ (\sum_r \tilde{u}_r C_r)|\tilde{x}_k\rangle \end{pmatrix} \right\|^2 \\
&= \sum_{j=1}^k \langle \tilde{x}_j|(\sum_r \tilde{u}_r C_r)^\dagger (\sum_r \tilde{u}_r C_r)|\tilde{x}_j\rangle \\
&\ge \min W_k\left((\sum_r \tilde{u}_r C_r)^\dagger (\sum_r \tilde{u}_r C_r) \right).
\end{aligned}$$

Similarly, we can choose $|\hat{\mathbf{x}}\rangle \in \Gamma_{n,k}$ so that $\tilde{D}_{\hat{\mathbf{x}}}$ has the largest maximum singular value $\sigma_1(\tilde{D}_{\hat{\mathbf{x}}})$ among all choice of $|\mathbf{x}\rangle \in \Gamma_{n,k}$. Then

$$\|\langle \mathbf{y}|\tilde{D}_{\mathbf{x}}\| \le \sigma_1(\tilde{D}_{\mathbf{x}}) \le \hat{\sigma}_1(\tilde{D}_{\hat{\mathbf{x}}}).$$

Moreover, there is a unit vector $|\hat{v}\rangle = (\hat{v}_1, \ldots, \hat{v}_q)^t \in \mathbf{C}^q$ such that

$$\begin{aligned}
\max W_k\left((\sum_s v_s D_s)^\dagger (\sum_s v_s D_s) \right) &\ge \sum_{j=1}^k \langle \hat{x}_j|(\sum_s \hat{v}_s D_s)^\dagger (\sum_s \hat{v}_s D_s)|\hat{x}_j\rangle \\
&= \left\| \begin{pmatrix} (\sum_s \hat{v}_s D_s)|\hat{x}_1\rangle \\ \vdots \\ (\sum_s \hat{v}_s D_s)|\hat{x}_k\rangle \end{pmatrix} \right\|^2 = \|\tilde{D}_{\hat{\mathbf{x}}}|\hat{v}\rangle\|^2 = (\sigma_1(\tilde{D}_{\hat{\mathbf{x}}}))^2.
\end{aligned}$$

By our assumption, we have $\sigma_p(\tilde{C}_{\tilde{\mathbf{x}}}) \ge \sigma_1(\tilde{D}_{\hat{\mathbf{x}}})$, and hence

$$\|\langle y|\tilde{C}_{\mathbf{x}}\| \ge \sigma_p(\tilde{C}_{\tilde{\mathbf{x}}}) \ge \sigma_1(\tilde{D}_{\hat{\mathbf{x}}}) \ge \|\langle y|\tilde{D}_{\mathbf{x}}\|.$$

The desired conclusion follows. □

Remark Note that, in the above proof, if there is $|\mathbf{x}\rangle \in \Gamma_{n,k}$ such that $\tilde{C}_{\mathbf{x}} = 0$, then

$$\min\left\{ W_k\left((\sum_r u_r C_r)^\dagger (\sum_r u_r C_r) \right) : |u\rangle = (u_1, \ldots, u_p)^t \in \mathbf{C}^p,\ \langle u|u\rangle = 1 \right\} = 0.$$

On the other hand, if

$$\min\left\{ W_k\left((\sum_r u_r C_r)^\dagger (\sum_r u_r C_r) \right) : |u\rangle = (u_1, \ldots, u_p)^t \in \mathbf{C}^p,\ \langle u|u\rangle = 1 \right\} > 0,$$

then $\tilde{C}_{\mathbf{x}}$ has rank kp for all $|\mathbf{x}\rangle \in \Gamma_{n,k}$.

4. Improvement of some results of Chruściński and Kossakowski

For $X \in M_{m,n}$ and $k \leq \min\{m,n\}$, let $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_{\min\{m,n\}}(X)$ be the singular values of X and define

$$\|X\|_k = \left\{\sum_{j=1}^{k} \sigma_j(X)^2\right\}^{1/2}.$$

Let $\{F_j\}_{j=1}^{mn} \subseteq M_{m,n}$, and define $\phi : M_n \to M_m$ by

$$L(X) = L_1(X) - L_2(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger - \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger, \quad \gamma_1, \ldots, \gamma_{mn} \geq 0.$$

Assume that

(C) *There is / for any orthonormal basis* $\{|x_1\rangle, |x_2\rangle, \ldots, |x_n\rangle\}$ *for* $\mathbf{C}^n$ *such that*

$$\left\{P_k = \sum_{i,j=1}^{n} |x_i\rangle\langle x_j| \otimes F_k|x_i\rangle\langle x_j|F_k^\dagger : 1 \leq k \leq mn\right\}$$

is an a set of mutually orthogonal set of rank one matrices.

It was proved in [6] that if (**C**) holds, and if $\sum_{j=p+1}^{mn} \|F_j\|_k^2 < 1$ and

$$\sum_{i,j=1}^{n} E_{ij} \otimes L_1(E_{ij}) \geq \frac{\sum_{j=p+1}^{mn} \|F_j\|_k^2}{1 - \sum_{j=p+1}^{mn} \|F_j\|_k^2} \left(I_n \otimes I_m - \sum_{j=p+1}^{mn} P_j\right),$$

then ϕ is k-positive. In [7], the authors stated the result using the assumption that $\{F_j\}_{j=1}^{mn} \subseteq M_{m,n}$ is an orthonormal set using the inner product $(X,Y) = \mathrm{tr}\,(XY^\dagger)$ in M_{mn} instead of condition (**C**). By the results in the previous section, we can refine and improve the results in [6, 7]. We first show that the above two conditions are equivalent.

Proposition 4.1 *Suppose* $\{F_1, \ldots, F_{mn}\} \subseteq M_{m,n}$. *The following two conditions are equivalent.*

(a) $\{F_1, \ldots, F_{mn}\}$ *is an orthonormal set, i.e.* $\mathrm{tr}\,(F_r^\dagger F_s) = \delta_{rs}$ *for* $r, s = 1, \ldots, mn$.

(b) *There is / for any orthonormal basis* $\{|x_1\rangle, \ldots, |x_n\rangle\}$ *for* $\mathbf{C}^n$,

$$\left\{\sum_{i,j=1}^{n} |x_i\rangle\langle x_j| \otimes F_r|x_i\rangle\langle x_j|F_r^\dagger : 1 \leq r \leq mn\right\}$$

is a set of mutually orthogonal rank one projections in M_{mn}.

Furthermore, if (a) or (b) holds, then we have

$$\sum_j F_j F_j^\dagger = nI_m \text{ and } \sum_j F_j^\dagger F_j = mI_n\,.$$

Proof. Suppose $F_1, \ldots, F_{mn} \in M_{m,n}$ and $\{|x_1\rangle, \ldots, |x_n\rangle\}$ is an orthonormal basis in $\mathbf{C}^n$. Define

$$P_r = \sum_{i,j=1}^n |x_i\rangle\langle x_j| \otimes F_r|x_i\rangle\langle x_j|F_r^\dagger \quad \text{for } r = 1, \ldots, mn.$$

Then for any $r, s = 1, \ldots, mn$, we have

$$\begin{aligned}
P_r P_s &= \left(\sum_{i,j=1}^n |x_i\rangle\langle x_j| \otimes F_r|x_i\rangle\langle x_j|F_r^\dagger\right)\left(\sum_{k,\ell=1}^n |x_k\rangle\langle x_\ell| \otimes F_s|x_k\rangle\langle x_\ell|F_s^\dagger\right) \\
&= \sum_{i,j,k,\ell=1}^n |x_i\rangle\langle x_j|x_k\rangle\langle x_\ell| \otimes F_r|x_i\rangle\langle x_j|F_r^\dagger F_s|x_k\rangle\langle x_\ell|F_s^\dagger \\
&= \left(\sum_{j,k=1}^n \langle x_j|x_k\rangle \cdot \langle x_j|F_r^\dagger F_s|x_k\rangle\right)\left(\sum_{i,\ell=1}^n |x_i\rangle\langle x_\ell| \otimes F_r|x_i\rangle\langle x_\ell|F_s^\dagger\right) \\
&= \left(\sum_{j=1}^n \langle x_j|F_r^\dagger F_s|x_j\rangle\right)\left(\sum_{i,\ell=1}^n |x_i\rangle\langle x_\ell| \otimes F_r|x_i\rangle\langle x_\ell|F_s^\dagger\right) \\
&= \operatorname{tr}(F_r^\dagger F_s)\left(\sum_{i,\ell=1}^n |x_i\rangle\langle x_\ell| \otimes F_r|x_i\rangle\langle x_\ell|F_s^\dagger\right).
\end{aligned}$$

Therefore, the implication (a) $\Rightarrow$ (b) holds. Now by taking the trace on both sides of the equation,

$$\begin{aligned}
\operatorname{tr}(P_r P_s) &= \operatorname{tr}(F_r^\dagger F_s) \cdot \operatorname{tr}\left(\sum_{i,\ell=1}^n |x_i\rangle\langle x_\ell| \otimes F_r|x_i\rangle\langle x_\ell|F_s^\dagger\right) \\
&= \operatorname{tr}(F_r^\dagger F_s) \cdot \left(\sum_{i,\ell=1}^n \langle x_\ell|x_i\rangle \cdot \langle x_\ell|F_s^\dagger F_r|x_i\rangle\right) \\
&= \operatorname{tr}(F_r^\dagger F_s) \cdot \left(\sum_{i=1}^n \langle x_i|F_s^\dagger F_r|x_i\rangle\right) \\
&= \operatorname{tr}(F_r^\dagger F_s) \cdot \operatorname{tr}(F_s^\dagger F_r) = |\operatorname{tr}(F_r^\dagger F_s)|^2.
\end{aligned}$$

Hence, (b) $\Rightarrow$ (a) followed by the above equlity.

Finally suppose (a) holds. We have

$$\operatorname{tr}(F_i F_j^\dagger) = \operatorname{tr}(F_j^\dagger F_i) = \delta_{ij}, \text{ for all } 1 \le j \le mn.$$

For $1 \le j \le mn$ and $1 \le r \le m$, let f_j^r be the r^{th} row of F_j. From $\operatorname{tr} F_i F_j^\dagger = \delta_{ij}$, we can form a unitary matrix $U \in M_{mn}$ with j^{th} row $u_j = \left[\, f_j^1 \mid \cdots \mid f_j^m \,\right]$. Since $U^\dagger U = I_{mn}$, for $1 \le r,\ s \le m$, we have $\sum_{j=1}^{mn} (f_j^s)^\dagger f_j^r = \delta_{rs} I_n$. Consider $R = \sum_{j=1}^{mn} F_j F_j^\dagger \in M_m$. The (r,s)-th entry of R is equal to

$$\sum_{j=1}^{mn} f_j^r (f_j^s)^\dagger = \operatorname{tr}\left(\sum_{j=1}^{mn} f_j^r (f_j^s)^\dagger\right) = \operatorname{tr}\left(\sum_{j=1}^{mn} (f_j^s)^\dagger f_j^r\right) = \operatorname{tr}(\delta_{rs} I_n) = n\delta_{rs}\,.$$

Therefore, $\sum_j F_j F_j^\dagger = nI_m$. Similarly, by replacing F_j with $F_j^\dagger$, $\sum_j F_j^\dagger F_j = mI_n$ follows from the fact that $\operatorname{tr} F_j^\dagger F_i = \delta_{ij}$ for all $1 \le i, j \le mn$. □

Proposition 4.2 *Suppose $\{F_j : 1 \le j \le mn\}$ is an orthonormal basis of $M_{m,n}$ and $L : M_n \to M_m$ has the form*

$$L(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger - \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger, \quad \gamma_1, \ldots, \gamma_{mn} \ge 0.$$

Assume that $1 \le k \le \min\{m,n\}$ and $\xi_k = 1 - \sum_{j=p+1}^{mn} \|F_j\|_k^2 > 0$.

(a) *If*

$$\gamma_i \ge \xi_k^{-1}\left(\sum_{j=p+1}^{mn} \gamma_j \|F_j\|_k^2\right) \qquad \text{for all} \quad i = 1, \ldots, p,$$

then L is k-positive.

(b) *If $p = mn - 1$ and*

$$\gamma_i < \xi_k^{-1} \gamma_{mn} \|F_{mn}\|_k^2 \qquad \text{for all} \quad i = 1, \ldots, mn-1,$$

then L is not k-positive.

Proof. Denote by $\Gamma_{n,k}$ the set of vectors $|\mathbf{x}\rangle$ such that $|\mathbf{x}\rangle = \begin{pmatrix} |x_1\rangle \\ \vdots \\ |x_k\rangle \end{pmatrix}$ where $\{|x_1\rangle, \ldots, |x_k\rangle\}$ is an orthonormal set in $\mathbf{C}^n$. We show that $I_k \otimes L(|\mathbf{x}\rangle\langle\mathbf{x}|)$ is positive semidefinite for any $|\mathbf{x}\rangle \in \Gamma_{n,k}$. We may extend $|\mathbf{x}\rangle$ to $|\tilde{\mathbf{x}}\rangle \in \Gamma_{n,n}$ with $|\tilde{\mathbf{x}}\rangle = \begin{pmatrix} |x_1\rangle \\ \vdots \\ |x_n\rangle \end{pmatrix}$ such that $\{|x_1\rangle, \ldots, |x_n\rangle\}$ is an orthonormal basis for $\mathbf{C}^n$. By Proposition 4.1(b),

$$\sum_{r=1}^{mn} (F_r |x_i\rangle\langle x_j| F_r^\dagger)_{1 \le i,j \le n} = I_{mn}.$$

Focusing on the leading $mk \times mk$ principal submatrix, we have

$$\sum_{r=1}^{p}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} = I_{mk} - \sum_{r=p+1}^{mn}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k}.$$

Note that

$$\begin{aligned}\mathrm{tr}\left(\sum_{r=p+1}^{mn}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k}\right) &= \sum_{j=1}^{k}\langle x_j|\left(\sum_{r=p+1}^{mn}\gamma_r F_r^\dagger F_r\right)|x_j\rangle \\ &\le \sum_{j=1}^{k}\sum_{r=p+1}^{mn}\gamma_r\lambda_j(F_r^\dagger F_r) = \gamma\end{aligned}$$

with $\gamma = \sum_{r=p+1}^{mn}\gamma_r\|F_r\|_k^2$, where $\lambda_j(A)$ denotes the j-th largest eigenvalue of a Hermitian matrix A. Thus,

$$\sum_{r=p+1}^{mn}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} \le \sum_{r=p+1}^{mn}\|F_r\|_k^2 I_{mk} \quad \text{and} \quad \sum_{r=p+1}^{mn}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} \le \gamma I_{mk}.$$

(a) If $\gamma\xi_k^{-1} \le \gamma_i$ for each $i = 1, \dots, p$, then we have

$$\begin{aligned}&\sum_{r=p+1}^{mn}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} \le \gamma I_{mk} = \gamma\xi_k^{-1}\left(I_{mk} - \left(\sum_{r=p+1}^{mn}\|F_r\|_k^2\right)I_{mk}\right) \\ &\le \gamma\xi_k^{-1}\left(I_{mk} - \sum_{r=p+1}^{mn}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k}\right) = \gamma\xi_k^{-1}\sum_{r=1}^{p}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} \\ &\le \sum_{r=1}^{p}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k}.\end{aligned}$$

Then L is k-positive by Proposition 3.1.

(b) Suppose that the hypothesis of (b) holds. We can choose $|x_1\rangle, \dots, |x_k\rangle$ in $\mathbf{C}^n$ so that

$$\mathrm{tr}\ (F_{mn}|x_i\rangle\langle x_j|F_{mn}^\dagger)_{1\le i,j\le k} = \|F_{mn}\|_k^2,$$

i.e., the rank one matrix $\gamma_{mn}(F_{mn}|x_i\rangle\langle x_j|F_{mn}^\dagger)_{1\le i,j\le k}$ has a nonzero eigenvalue $\gamma_{mn}\|F_{mn}\|_k^2$. Now,

$$\sum_{r=1}^{mn-1}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} < \gamma_{mn}\xi_k^{-1}\left(\sum_{r=1}^{mn-1}(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k}\right) \le \gamma_{mn}I_{mk}.$$

Thus, the matrix

$$\sum_{r=1}^{mn-1}\gamma_r(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le k} - \gamma_{mn}(F_{mn}|x_i\rangle\langle x_j|F_{mn}^\dagger)_{1\le i,j\le k}$$

has a negative eigenvalue. The result follows. □

Note that we only need to focus on $mk \times mk$ matrices in our proof. But there is no harm to apply the arguments to the extended matrices $(F_r|x_i\rangle\langle x_j|F_r^\dagger)_{1\le i,j\le n}$ to get the results in [6] and [7]. Actually, using the same proof and the concept of the k-numerical range, we can improve part (a) of the above proposition to the following.

Proposition 4.3 *Suppose that* $\{F_j : 1 \le j \le mn\}$ *is an orthonormal basis of* $M_{m,n}$ *and* $L : M_n \to M_m$ *has the form*

$$L(X) = \sum_{j=1}^{p} \gamma_j F_j X F_j^\dagger - \sum_{j=p+1}^{mn} \gamma_j F_j X F_j^\dagger, \quad \gamma_1, \ldots, \gamma_{mn} \ge 0.$$

Assume that $1 \le k \le \min\{m, n\}$ *and* $\tilde{\xi}_k = 1 - \max W_k(\sum_{j=p+1}^{mn} F_j^\dagger F_j) > 0$. *If*

$$\gamma_i \ge \tilde{\xi}_k^{-1} \max W_k \left(\sum_{j=p+1}^{mn} \gamma_j F_j^\dagger F_j \right), \qquad i = 1, \ldots, p,$$

then L *is* k*-positive.*

5. Criteria for k-positivity of D-type linear maps

In this section, we consider linear maps $L : M_n \to M_n$ of the form

$$(a_{ij}) \mapsto \operatorname{diag}(f_1, \ldots, f_n) - (a_{ij}) \qquad \text{with} \quad (f_1, \ldots, f_n) = (a_{11}, \ldots, a_{nn})D \tag{5}$$

for an $n \times n$ nonnegative matrix $D = (d_{ij})$. Such kind of maps will be called D-type linear maps. The question of when a D-type map is positive was studied intensively by many authors and applied in quantum information theory to detect entangled states and construct entanglement witnesses. For example, if $D = (n-1)I_n + E_{12} + \cdots + E_{n-1,n} + E_{n,1}$, we get a positive map which is not completely positive. This can be viewed as a generalization of the Choi map in [4].

In the following, we present a necessary and sufficient criterion of D-type linear map to be k-positive.

Proposition 5.1 *Suppose* $L : M_n \to M_n$ *is a* D*-type map (5) for an* $n \times n$ *nonnegative matrix* $D = (d_{ij})$. *The following conditions are equivalent.*

(a) L *is* k*-positive.*

(b) $d_{ii} > 0$ *for all* $i = 1, \ldots, n$. *For any unit vector* $|\mathbf{u}\rangle = (u_1, \ldots, u_{nk})^t \in \mathbf{C}^{nk}$, *let*

$$[\mathbf{u}] = (\, |\hat{\mathbf{u}}_1\rangle \quad \cdots \quad |\hat{\mathbf{u}}_n\rangle \,) = \begin{pmatrix} u_1 & \cdots & u_n \\ u_{n+1} & \cdots & u_{2n} \\ \vdots & \ddots & \vdots \\ u_{(k-1)n+1} & \cdots & u_{kn} \end{pmatrix}.$$

where $|\hat{\mathbf{u}}_\ell\rangle$ is the ℓ-th column of $[\mathbf{u}]$. Set $D_\ell = [\mathbf{u}]\mathrm{diag}\,(d_{1,\ell},\ldots,d_{n,\ell})[\mathbf{u}]^\dagger$, Then

$$\sum_{\ell=1}^{n} \langle \hat{\mathbf{u}}_\ell | D_\ell^{[-1]} | \hat{\mathbf{u}}_\ell \rangle \le 1,$$

where $X^{[-1]}$ is the Moore-Penrose generalized inverse of X.

Proof. For any unit vector $|\mathbf{u}\rangle \in \mathbf{C}^{nk}$, write

$$|\mathbf{u}\rangle = \sum_{j=1}^{n} |\hat{\mathbf{u}}_j\rangle \otimes |\hat{e}_j\rangle = \sum_{j=1}^{k} |e_j\rangle \otimes |\mathbf{u}_j\rangle,$$

where $\{|e_1\rangle,\ldots,|e_k\rangle\}$ and $\{|\hat{e}_1\rangle,\ldots,|\hat{e}_n\rangle\}$ are the standard base of $\mathbf{C}^k$ and $\mathbf{C}^n$ respectively. Here, $|\hat{\mathbf{u}}_j\rangle$ is the j-th column of $[\mathbf{u}]$ and $|\mathbf{u}_j\rangle$ is the transpose of the j-th row of $[\mathbf{u}]$. Define

$$F_{ij} = L(|\mathbf{u}_i\rangle\langle\mathbf{u}_j|) + |\mathbf{u}_i\rangle\langle\mathbf{u}_j|,$$

which is a diagonal matrix with the ℓ-th diagonal entry equal to $\langle\mathbf{u}_j|\mathrm{diag}\,(d_{1,\ell},\ldots,d_{n,\ell})|\mathbf{u}_i\rangle$. By Proposition 2.1, L is k-positive if and only if the $nk \times nk$ matrix

$$(I_k \otimes L)(|\mathbf{u}\rangle\langle\mathbf{u}|) = \sum_{i,j=1}^{k} |e_i\rangle\langle e_j| \otimes (F_{ij} - |\mathbf{u}_i\rangle\langle\mathbf{u}_j|) = \left(\sum_{i,j=1}^{k} |e_i\rangle\langle e_j| \otimes F_{ij}\right) - |\mathbf{u}\rangle\langle\mathbf{u}|$$

is positive semi-definite. Notice that the above matrix is permutationally similar to

$$\sum_{i,j=1}^{k} (F_{ij} - |\mathbf{u}_i\rangle\langle\mathbf{u}_j|) \otimes |e_i\rangle\langle e_j| = \left(\sum_{i,j=1}^{k} F_{ij} \otimes |e_i\rangle\langle e_j|\right) - |\hat{\mathbf{u}}\rangle\langle\hat{\mathbf{u}}|,$$

where $|\hat{\mathbf{u}}\rangle = \sum_{j=1}^{n} |\hat{e}_j\rangle \otimes |\mathbf{u}_j\rangle$. Direct computation shows that

$$\sum_{i,j=1}^{k} F_{ij} \otimes |e_i\rangle\langle e_j| = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

Therefore, the condition is equivalent to the following.

(c) $|\hat{\mathbf{u}}\rangle$ lies in the range of $D_1 \oplus \cdots D_n$ and $\|(D_1 \oplus \cdots \oplus D_n)^{1/2})^{[-1]}|\hat{\mathbf{u}}\rangle\| \le 1$.

Finally, $|\hat{\mathbf{u}}_\ell\rangle$ lies in the range of D_ℓ for all unit vector $|\mathbf{u}\rangle$ and $\ell = 1,\ldots,n$ if and only if $d_{\ell\ell} > 0$ for all $\ell = 1,\ldots,n$ and the norm in (c) is the same as the sum as stated in (b). Therefore, the result follows. □

Proposition 5.1 is particular useful when $k = 1$.

Corollary 5.2 *Let $L : M_n \to M_m$ be a D-type map of the form (5) with $D = (d_{ij})$. For $u = (u_1, u_2, \ldots, u_n)^t \in \mathbf{C}^n$, let $f_j(u) = \sum_{i=1}^n d_{ij}|u_i|^2$. Then, L is positive if and only if any one of the following equivalent conditions hold*

(1) *$d_{ii} > 0$ for all $i = 1, \ldots, n$ and $\sum_{u_j \neq 0} \frac{|u_j|^2}{f_j(u)} \leq 1$ for every unit vector $|\mathbf{u}\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbf{C}^n$.*

(2) *$d_{ii} > 0$ for all $i = 1, \ldots, n$ and $\sum_{j=1}^n \frac{|u_j|^2}{f_j(u)} \leq 1$ for every vector $|\mathbf{u}\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbf{C}^n$ with $u_i \neq 0$ for all $i = 1, \ldots, n$.*

Proof. For $k = 1$, (1) is equivalent to condition (b) in Proposition 5.1. (2) is equivalent to (1) because $\frac{|u_j|^2}{f_j(u)}$ is continuous and homogeneous in u. □

6. Constructing D-type positive maps

In this section, we discuss how to construct D-type positive linear maps using the results in previous sections.

The following example is well-known. Here we give a different proof by applying Proposition 2.1. Notice that the map is a D-type map with all entries of D being γ.

Example 6.1 *For $\gamma \geq 0$, define $L_\gamma : M_n \to M_n$ by*

$$L_\gamma(A) = \gamma(\operatorname{tr} A)I_n - A.$$

Then for any $k \in \{1, \ldots, n\}$, L_γ is k-positive if and only if $\gamma \geq k$.

Proof. For any $|x\rangle = \sum_{j=1}^k |e_j\rangle \otimes |x_j\rangle \in \mathbf{C}^{nk}$ with an orthonormal set $\{|x_j\rangle : 1 \leq j \leq k\}$ in $\mathbf{C}^n$,

$$\begin{aligned}\sum_{i,j=1}^k E_{ij} \otimes L_\gamma(|x_i\rangle\langle x_j|) &= \sum_{i,j=1}^k E_{ij} \otimes (\gamma(\operatorname{tr} |x_i\rangle\langle x_j|)I_n - |x_i\rangle\langle x_j|) \\ &= \sum_{i,j=1}^k E_{ij} \otimes (\gamma(\operatorname{tr} \langle x_j|x_i\rangle)I_n - |x_i\rangle\langle x_j|) = \gamma I_{kn} - |x\rangle\langle x|.\end{aligned}$$

Since $|x\rangle\langle x|$ is a rank one hermitian matrix with trace k, by Proposition 2.1, L_γ is k-positive if and only if $\gamma \geq k$. □

Recall that a permutation π of $(i_1, \ldots, i_\ell)$ is an ℓ-cycle if $\pi(i_j) = i_{j+1}$ for $j = 1, \ldots, \ell - 1$ and $\pi(i_\ell) = i_1$. Note that every permutation π of $(1, \ldots, n)$ has a disjoint cycle decomposition $\pi = (\pi_1)(\pi_2)\cdots(\pi_r)$, that is, there exists a set $\{F_s\}_{s=1}^r$ of disjoint cycles of π with $\cup_{s=1}^r F_s = \{1, 2, \ldots, n\}$ such that $\pi_s = \pi|_{F_s}$ and $\pi(i) = \pi_s(i)$ whenever $i \in F_s$. We have the following.

Proposition 6.2 *Suppose π is a permutation of $(1, 2, \ldots, n)$ with disjoint cycle decomposition $\pi_1 \cdots \pi_r$ such that the maximum length of π_i is equal to $\ell > 1$ and $P_\pi = \left(\delta_{i\pi(j)}\right)$ is the permutation matrix associated with π. For $t \geq 0$, let $\Phi_{t,\pi} : M_n \to M_n$ be the D-type map of the form (5) with $D = (n-t)I_n + tP_\pi$. Then $\Phi_{t,\pi}$ is positive if and only if $t \leq \frac{n}{\ell}$.*

Proof. It is easily checked that for $0 \leq t \leq 1$, the function

$$g(r_1, r_2, \ldots, r_s) = \sum_{i=1}^{s} \frac{1}{s - t + tr_i} \leq 1 \quad \text{for all } r_i > 0 \text{ and } r_1 r_2 \cdots r_s = 1, \tag{6}$$

and the function g attains the maximum 1 when $r_1 = \cdots = r_s = 1$.

Suppose $0 \leq t \leq \frac{n}{\ell}$. We are going to use condition (2) in Corollary 5.2 to show that $\Phi_{t,\pi}$ is positive. For any vector $|u\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbf{C}^n$, with $u_i \neq 0$ for all $i = 1, \ldots, n$, we have $f_i(u) = (n-t)|u_i|^2 + t|u_{\pi(i)}|^2$. So, by Corollary 5.2, $\Phi_{t,\pi}$ is positive if

$$f(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \frac{|u_i|^2}{(n-t)|u_i|^2 + t|u_{\pi(i)}|^2} \leq 1 \tag{7}$$

for all vector $|u\rangle = (u_1, u_2, \ldots, u_n)^t$ with nonzero entries.

Suppose π is a product of r disjoint cycles, that is, $\pi = (\pi_1)(\pi_2) \cdots (\pi_r)$. Let F_j be the set of indices corresponding to the cycle π_j and ℓ_j denote the number of elements in F_j for $j = 1, \ldots, r$. Then $\ell = \max\{\ell_1, \ldots, \ell_r\}$ and $\sum_j \ell_j = n$. For any vector $|u\rangle = (u_1, u_2, \ldots, u_n)^t \in \mathbf{C}^n$, with $u_i \neq 0$ for all $i = 1, \ldots, n$, we have $\prod_{i \in F_j} \frac{|u_{\pi_j(i)}|^2}{|u_i|^2} = 1$. It follows that

$$\begin{aligned}
f(u_1, u_2, \ldots, u_n) &= \sum_{i=1}^{n} \frac{|u_i|^2}{(n-t)|u_i|^2 + t|u_{\pi(i)}|^2} \\
&= \sum_{j=1}^{r} \sum_{i \in F_j} \frac{|u_i|^2}{(n-t)|u_i|^2 + t|u_{\pi_j(i)}|^2} \\
&= \sum_{j=1}^{r} \frac{\ell_j}{n} \sum_{i \in F_j} \frac{1}{\ell_j - \frac{\ell_j}{n} t + \frac{\ell_j}{n} t \frac{|u_{\pi_j(i)}|^2}{|u_i|^2}} \\
&\leq \sum_{j=1}^{r} \frac{\ell_j}{n} \cdot 1 = 1,
\end{aligned}$$

whenever $0 \leq \frac{\ell_j}{n} t \leq 1$ for all $1 \leq j \leq r$ by (6), or equivalently, $0 \leq t \leq \frac{n}{\ell_j} \leq \frac{n}{\ell}$. Therefore, (7) holds.

Conversely, suppose $t > \frac{n}{\ell}$. Let $\pi = (\pi_1)(\pi_2) \cdots (\pi_r)$ be a decomposition of π into disjoint cycles. Without loss of generality, we may assume that $\ell_1 = \ell \geq \ell_j$ for all

$j = 2, \ldots, r$, and π_1 is a cycle on $(1, 2, \ldots, \ell)$. Let $u_i = \epsilon^{\frac{i}{2}}$, where $0 < \epsilon < 1 - \frac{n}{\ell t}$ for $i = 1, \ldots, \ell$ and $u_i = 1$ for $\ell + 1 \leq i \leq n$. Then we have

$$\begin{aligned} f(u_1, u_2, \ldots, u_n) &= \sum_{i=1}^{\ell-1} \frac{1}{(n-t)+t\epsilon} + \frac{1}{(n-t)+\frac{t}{\epsilon^{\ell-1}}} + \sum_{i=\ell+1}^{n} \frac{1}{(n-t)+t} \\ &\geq \frac{\ell-1}{(n-t)+t\epsilon} + \frac{n-\ell}{n} \\ &> \frac{\ell-1}{(n-t)+t\left(1-\frac{n}{\ell t}\right)} + \frac{n-\ell}{n} \\ &= \frac{\ell-1}{n-\frac{n}{\ell}} + \frac{n-\ell}{n} = \frac{\ell}{n} + \frac{n-\ell}{n} = 1, \end{aligned}$$

which implies $\Phi_{t,\pi}$ is not positive. □

Next, we consider a general map Λ_D of the form (5).

Proposition 6.3 *Let $\Lambda_D : M_n \to M_n$ have the form (5) for a nonnegative matrix $D = (d_{ij})$ with all row sum and column sum equal to n. Then Λ_D is positive if $d_{ii} \geq (n-1)$ for all $i = 1, \ldots, n$. Moreover, the following conditions are equivalent.*

(a) *Λ_D is completely positive.* (b) *Λ_D is 2-positive.* (c) *$D = nI_n$.*

Proof. Suppose $d_{ii} \geq n-1$ for all $i = 1, \ldots, n$. Then $D = (n-1)I + S$ for a doubly stochastic matrix S, which is a convex combination of permutation matrices (e.g., see [11, Theorem 8.7.1, pp. 527]). We may represent S as

$$S = \sum_{i=1}^{m} p_i P_{\pi_i}$$

for some permutations $\pi_1, \pi_2, \ldots, \pi_m$ of $\{1, 2, \ldots, n\}$ and positive scalars p_i with $\sum_{i=1}^{m} p_i = 1$. Let $S_i = (n-1)I_n + P_{\pi_i}$ and Λ_{S_i} be the linear map of the form as in (5). By Proposition 6.2, Λ_{S_i} is a positive map. Thus, Λ_D is a convex combination of positive maps, and is therefore positive.

Next, we prove the three equivalent conditions. The implication (a) $\Rightarrow$ (b) is clear. For (c) $\Rightarrow$ (a), it is well known and easy to check, say, by considering the Choi matrix, that Λ_D is completely positive if $D = nI_n$.

It remains to prove (b) $\Rightarrow$ (c). Suppose $D \neq nI_n$. Then $d_{ii} < n$ for some i. Without loss of generality, we assume that $i = 1$. Let $|\mathbf{u}\rangle \in \mathbf{C}^{2n}$ be such that

$$[\mathbf{u}] = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix}.$$

Then

$$D_\ell = \frac{1}{n}\begin{pmatrix} d_{1\ell} & 0 \\ 0 & n - d_{1\ell} \end{pmatrix}, \quad \ell = 1, \ldots, n.$$

As $n > d_{11}$,

$$\langle \hat{\mathbf{u}}_1 | D_1^{[-1]} | \hat{\mathbf{u}}_1 \rangle = \frac{1}{d_{11}} > \frac{1}{n} \quad \text{and} \quad \langle \hat{\mathbf{u}}_\ell | D_\ell^{[-1]} | \hat{\mathbf{u}}_\ell \rangle = \frac{1}{n - d_{1\ell}} \geq \frac{1}{n} \quad \text{for } \ell = 2, \ldots, n.$$

Hence,

$$\sum_{\ell=1}^{n} \langle \hat{\mathbf{u}}_\ell | D_\ell^{[-1]} | \hat{\mathbf{u}}_\ell \rangle > 1,$$

and Λ_D is not 2-positive by Proposition 5.1. □

In [22], the positive map Λ_D with $D = (n-1)I_n + P$ for a permutation matrix P was considered, and the special case when P is a length n-cycle was discussed in details. By Propositions 6.2 and 6.3, we have the following corollary.

Corollary 6.4 *Let $\Lambda_D : M_n \to M_n$ be a D-type map of the form (5) with $D = (n-1)I_n + P$ for a permutation matrix P. Then Λ_D is positive. Moreover, the following are equivalent.*

(a) *Λ_D is completely positive.* (b) *Λ_D is 2-positive.* (c) *$D = nI_n$.*

The condition $d_{ii} \geq n-1$ for each i is not necessary for Λ_D in Proposition 6.3 to be positive as seen below.

Example 6.5 *Let* $D = \begin{pmatrix} 1.35 & 1 & 0.65 \\ 0.65 & 1.35 & 1 \\ 1 & 0.65 & 1.35 \end{pmatrix}$. *Here, $d_{ii} < 2 = 3 - 1$. Direct computation shows that $\sum_{j=1}^{3} \frac{|u_j|^2}{f_j(u)} \leq 1$ for all $(u_1, u_2, u_3) \in \mathbf{C}^3$. Therefore, Λ_D is positive by Corollary 5.2.*

Example 6.6 *In Proposition 6.2, let $0 \leq t \leq 1$ and $D = (d_{ij}) = (n-t)I_n + tS$, where* $S = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_n & s_1 & \cdots & s_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & s_3 & \cdots & s_1 \end{pmatrix}$ *with $s_i \geq 0$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} s_i = 1$. Define $\Lambda_D : M_n \to M_n$ by*

$$\Lambda_D((a_{ij})) = \begin{pmatrix} f_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & f_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & f_n \end{pmatrix},$$

where

$$
\begin{aligned}
f_1 &= (n-t-1+ts_1)a_{11} + ts_n a_{22} + ts_{n-1}a_{33} + \cdots + ts_2 a_{nn},\\
f_2 &= ts_2 a_{11} + (n-t-1+ts_1)a_{22} + ts_n a_{33} + \cdots + ts_3 a_{nn},\\
&\ \ \vdots\\
f_n &= ts_n a_{11} + ts_{n-1}a_{22} + ts_{n-2}a_{33} + \cdots + (n-t-1+ts_1)a_{nn}.
\end{aligned}
$$

By Proposition 6.2, the map Λ_D is positive.

Finally, we give an example which illustrates how to apply Proposition 6.2 to construct positive elementary operators for any dimension.

Example 6.7 *Let H and K be Hilbert spaces of dimension at least n, and let $\{|e_i\rangle\}_{i=1}^n$ and $\{|\hat{e}_j\rangle\}_{j=1}^n$ be any orthonormal sets of H and K, respectively. For any permutation $\pi \neq \mathrm{id}$ of $\{1, 2, \ldots, n\}$, let $l(\pi) = l \leq n$. Let $\Phi_{t,\pi} : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by*

$$
\Phi_{t,\pi}(A) = (n-t)\sum_{i=1}^{n} E_{ii}AE_{ii}^\dagger + t\sum_{i=1}^{n} E_{i,\pi(i)}AE_{i,\pi(i)}^\dagger - (\sum_{i=1}^{n} E_{ii})A(\sum_{i=1}^{n} E_{ii})^\dagger
$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |\hat{e}_j\rangle\langle e_i|$. Then $\Phi_{t,\pi}$ is positive if and only if $0 \leq t \leq \frac{n}{l}$.

In fact, for the case $\dim H = \dim K = n$, $\Phi_{t,\pi}$ is a D-type map of the form (5) with $D = (n-t)I + tP_\pi$ as discussed in Proposition 6.2.

7. Decomposable D-type positive maps

Decomposability of positive linear maps is a topic of particular importance in quantum information theory since it is related to the PPT states (that is, the states with positive partial transpose). In this section, we will give a new class of decomposable positive linear maps.

The following result is well known (see [13]).

Proposition 7.1 *Suppose $L : M_n \to M_m$ has the form (5). Then L is decomposable if and only if the Choi matrix $C(L)$ is a sum of two matrices C_1 and C_2 such that C_1 and the partial transpose of C_2 are positive semi-definite.*

In [22], it was shown that the linear maps $\Phi^{(k)} = \Phi_{1,\pi}$ with $\pi(i) = i + k \pmod{n}$ in Proposition 6.2 are indecomposable whenever either n is odd or $k \neq \frac{n}{2}$. It was asked in [22] that whether or not $\Phi^{(\frac{n}{2})}$ is decomposable when n is even. In this section, we will answer this question by showing that $\Phi^{(\frac{n}{2})}$ is decomposable. In fact, this is a special case of the following proposition as $(\pi)^2 = \mathrm{id}$.

Proposition 7.2 *Let π be a permutation of $\{1, 2, \ldots, n\}$. If $\pi^2 = \mathrm{id}$, then the positive linear map $\Phi_{1,\pi}$ in Proposition 6.2 is decomposable.*

Proof. For simplicity, denote $\Phi = \Phi_{1,\pi}$. Let F be the set of fixed points of π. Since $\Phi(E_{ii}) = (n-2)E_{ii} + E_{\pi(i),\pi(i)}$ and $\Phi(E_{ij}) = -E_{ij}$, the Choi matrix of Φ is

$$\begin{aligned} C(\Phi) &= \sum_{i=1}^{n}(n-2)E_{ii}\otimes E_{ii} + \sum_{i=1}^{n} E_{\pi(i),\pi(i)} \otimes E_{ii} - \sum_{i\neq j} E_{ij}\otimes E_{ij} \\ &= \sum_{i\in F}(n-1)E_{ii}\otimes E_{ii} + \sum_{i\notin F}(n-2)E_{ii}\otimes E_{ii} \\ &\quad - \sum_{i\neq j;\pi(i)\neq j} E_{ij}\otimes E_{ij} + \sum_{i\notin F} E_{\pi(i),\pi(i)}\otimes E_{ii} - \sum_{i\notin F} E_{i,\pi(i)}\otimes E_{i,\pi(i)}. \end{aligned}$$

Let

$$C_1 = \sum_{i\in F}(n-1)E_{ii}\otimes E_{ii} + \sum_{i\notin F}(n-2)E_{ii}\otimes E_{ii} - \sum_{i\neq j;\pi(i)\neq j} E_{ij}\otimes E_{ij}$$

and

$$C_2 = \sum_{i\notin F} E_{\pi(i),\pi(i)}\otimes E_{ii} - \sum_{i\notin F} E_{i,\pi(i)}\otimes E_{i,\pi(i)}.$$

Since $\pi^2 = \mathrm{id}$, the cardinal number of F^c must be even. Thus we have

$$C_2 = \sum_{i<\pi(i)} \left(E_{\pi(i),\pi(i)}\otimes E_{ii} + E_{ii}\otimes E_{\pi(i),\pi(i)} - E_{i,\pi(i)}\otimes E_{i,\pi(i)} - E_{\pi(i),i}\otimes E_{\pi(i),i}\right).$$

As

$$C_2^{\mathrm{T}_2} = \sum_{i<\pi(i)} \left(E_{\pi(i),\pi(i)}\otimes E_{ii} + E_{ii}\otimes E_{\pi(i),\pi(i)} - E_{i,\pi(i)}\otimes E_{\pi(i),i} - E_{\pi(i),i}\otimes E_{i,\pi(i)}\right) \geq 0,$$

we see that C_2 is PPT.

Observe that $C_1 \cong A \oplus 0$, where $A = (a_{ij}) \in M_n$ is a Hermitian matrix satisfying $a_{ii} = n-2$ or $n-1$, $a_{ij} = 0$ or -1 so that $\sum_{j=1}^{n} a_{ij} = 0$. It is easily seen from the strictly diagonal dominance theorem (Ref. [11, Theorem 6.1.10, pp. 349]) that A is positive semi-definite. So $C_1 \geq 0$, and by Proposition 7.1, Φ is decomposable. $\square$

8. Conclusion

Because k-positive linear maps are important in theory as well as applications, many researchers have been working on problems such as finding efficient criteria to determine k-positive maps and constructing k-positive maps with simple structure. In this paper, we present some existing and new criteria for k-positive maps. Using these criteria, we

are able to improve the results of other researchers. Moreover, new classes of k-positive maps are introduced, and the decomposability of the maps are discussed. These lead the answers of some open problems.

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